

Spontaneous Symmetry Breaking in Wormholes Spacetimes with Matter

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When bosonic matter in the form of a complex scalar field is added to Ellis wormholes, the phenomenon of spontaneous symmetry breaking is observed. Symmetric solutions possess full reflection symmetry with respect to the radial coordinate of the two asymptotically flat spacetime regions connected by the wormhole, whereas asymmetric solutions do not possess this symmetry. Depending on the size of the throat, at bifurcation points pairs of asymmetric solutions arise from or merge with the symmetric solutions. These asymmetric solutions are energetically favoured. When the backreaction of the boson field is taken into account, this phenomenon is retained. Moreover, in a certain region of the solution space both symmetric and asymmetric solutions exhibit a transition from single throat to double throat configurations.

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I. INTRODUCTION

Spontaneous symmetry breaking is a ubiquitous phenomenon in physics. Its wide range of applications includes, for instance, the Higgs mechanism in particle physics, which allows to give mass to the particles of the Standard Model, or the phase transitions of ferromagnets in solid state physics. Here we consider this phenomenon in gravity.

In particular, we consider an Ellis wormhole in General Relativity [1–8]. Such a wormhole connects two asymptotically flat spacetimes by a throat. In order to allow for the non-trivial topology in General Relativity a phantom field is included, i.e., a real scalar field with a reversed sign in front of its kinetic term in the action.

When symmetric boundary conditions are specified, the Ellis wormhole is reflection symmetric with respect to its throat. In suitable coordinates, the wormhole metric in $D = 4$ spacetime dimensions reads

$$ds^2 = -dt^2 + d\eta^2 + (\eta^2 + \eta_0^2) d\Omega^2. \quad (1)$$

Then the wormhole throat is located at $\eta = 0$, and the throat parameter η_0 characterizes the size of the throat. Of course, by choosing asymmetric boundary conditions also asymmetric wormhole solutions can be created, where the reflection symmetry $\eta \rightarrow -\eta$ no longer holds. However, here we are not interested in such an enforced symmetric breaking. Instead we would like to demonstrate that asymmetric wormholes can appear also for symmetric boundary conditions, when matter is included. Thus the symmetry breaking happens spontaneously, and energy considerations will tell us, that the asymmetric solutions are energetically preferred.

Wormholes immersed in matter have been studied before with various types of matter. Examples include nuclear matter [9–11] and bosonic matter [12]. However, most solutions studied so far were symmetric, and the asymmetric solutions were asymmetric by construction, because of an asymmetric choice of boundary conditions. Thus - to our knowledge - for wormhole solutions immersed in matter spontaneous symmetry breaking has not been observed before.

Recently, the static Ellis wormhole solution has been generalized to spacetimes with higher dimensions [13, 14]. Moreover, rotating generalizations of the Ellis wormhole in four and five dimensions have been found [14–16]. We will not address the question of rotation here. Instead we will consider non-rotating wormhole solutions in D dimensions immersed in bosonic matter, with a focus on five dimensions.

The bosonic matter consists of a complex scalar field with a self-interaction such that it allows for localized solutions already in a flat spacetime background, which correspond to non-topological solitons [17, 18] or Q -balls [19]. When

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gravity is coupled to this bosonic matter boson stars arise [18, 20], which can get very close to the black hole limit. In [12] in addition a phantom field was included, to obtain solutions with a non-trivial topology, i.e., Q -balls and boson stars harbouring wormholes at their core. However, all these solutions were reflection symmetric.

Here we study the emergence of the asymmetric solutions. Let us denote with \mathcal{M}_+ the part of the manifold with positive radial coordinate η , and with \mathcal{M}_- the part with negative η . Since we are dealing with configurations with two asymptotically flat regions, we obtain two values for the mass, M_{\pm} , as measured asymptotically in \mathcal{M}_{\pm} . These values can be read off the asymptotic behaviour of the metric. For symmetric solutions both masses agree. However, for asymmetric solutions the two masses can differ widely. We note, that the asymmetric solutions always come in pairs, since they are transformed into each other by the reflection transformation $\eta \rightarrow -\eta$. Consequently, their two masses are simply interchanged.

To obtain the particle number in the case of Q -balls and boson stars, volume integrals are performed. For the solutions with non-trivial topology one can proceed analogously, when the solutions are symmetric. For the asymmetric solutions, however, the calculation of the particle number via such integrals can become ambiguous, since the inner boundary is not provided by symmetry. Therefore we here propose an unambiguous procedure to obtain the particle number. This is mandatory, since we need the particle number in order to demonstrate that the asymmetric solutions are energetically favourable.

The presence of the matter in the wormhole spacetime has further notable consequences. In particular, the back-reaction of the matter on the geometry can cause a drastic change of the geometry, giving rise to a transition from configurations with a single throat to configurations with a double throat and an equator in between. Such a transition is known to occur for symmetric wormholes with bosonic matter [12] and for other types of matter as well [21, 22].

This paper is organized as follows: In section II we present the theoretical setting for obtaining both symmetric and asymmetric configurations of wormholes immersed in bosonic matter in D dimensions. The numerical results are shown in section III, starting with the probe limit, where the spontaneous symmetry breaking is already observed. We then discuss in detail the five-dimensional families of symmetric and asymmetric solutions in the presence of gravity. Finally, we include a brief discussion of the asymmetric configurations in four dimensions, and show that the spontaneous symmetry breaking is present as well. We conclude in section V. In appendix A we briefly address our method of extracting the mass and the particle number for asymmetric solutions.

II. THEORETICAL SETTING

A. Action

We consider General Relativity with a minimally coupled complex scalar field Φ and phantom field Ψ in D spacetime dimensions. Besides the Einstein-Hilbert action with curvature scalar \mathcal{R} , coupling constant κ and metric determinant g , the action

$$S = \int \left[\frac{1}{2\kappa} \mathcal{R} + \mathcal{L}_{\text{ph}} + \mathcal{L}_{\text{bs}} \right] \sqrt{-g} \, d^D x \quad (2)$$

then contains the respective matter contributions, the Lagrangian \mathcal{L}_{ph} of the phantom field Ψ , and the Lagrangian \mathcal{L}_{bs} of the complex scalar field Φ . The kinetic term for the phantom field carries the reverse sign

$$\mathcal{L}_{\text{ph}} = \frac{1}{2} \partial_{\mu} \Psi \partial^{\mu} \Psi, \quad (3)$$

as compared to the kinetic term of the complex scalar field

$$\mathcal{L}_{\text{bs}} = -\frac{1}{2} g^{\mu\nu} (\partial_{\mu} \Phi^* \partial_{\nu} \Phi + \partial_{\nu} \Phi^* \partial_{\mu} \Phi) - U(|\Phi|). \quad (4)$$

The asterisk denotes complex conjugation, and U denotes the potential with the mass term and the self-interaction

$$U(|\Phi|) = \lambda |\Phi|^2 (|\Phi|^4 - c |\Phi|^2 + b). \quad (5)$$

The global minimum of the potential resides at $\Phi = 0$, where $U(0) = 0$, while a local minimum is found at some finite value of $|\Phi|$. The mass of the bosons $m_b = \sqrt{\lambda b}$ is specified by the quadratic term. The potential is chosen such that it allows for non-topological soliton solutions [17, 18] or Q -balls [19].

Variation of the action with respect to the metric leads to the Einstein equations

$$G_{\mu\nu} = \mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} = \kappa T_{\mu\nu} \quad (6)$$

with stress-energy tensor

$$T_{\mu\nu} = g_{\mu\nu}\mathcal{L}_M - 2\frac{\partial\mathcal{L}_M}{\partial g^{\mu\nu}}, \quad (7)$$

where we denoted by $\mathcal{L}_M = \mathcal{L}_{\text{ph}} + \mathcal{L}_{\text{bs}}$ the sum of the scalar field Lagrangians.

B. Ansätze

For the line element of the spherically symmetric solutions with a non-trivial topology we choose

$$ds^2 = -e^{(D-3)a}dt^2 + pe^{-a} [d\eta^2 + (\eta^2 + \eta_0^2)\Omega_{D-2}^2] . \quad (8)$$

Here $d\Omega_{D-2}^2$ denotes the metric on the unit $(D-2)$ -sphere, while a and p are functions of the radial coordinate η , which takes positive and negative values, i.e. $-\infty < \eta < \infty$. The two limits $\eta \rightarrow \pm\infty$ correspond to two distinct asymptotically flat regions, associated with \mathcal{M}_+ and \mathcal{M}_- , respectively. Note that in Eq. (8) we have introduced the parameter η_0 , which we will refer to as the throat parameter.

As for spherically symmetric Q -balls and boson stars, we parametrize the complex scalar field Φ via

$$\Phi = \phi(\eta) e^{i\omega_s t}, \quad (9)$$

where $\phi(\eta)$ is a real function, and ω_s denotes the boson frequency. The phantom field Ψ depends only on the radial coordinate,

$$\Psi = \psi(\eta) . \quad (10)$$

C. Einstein and Matter Field Equations

Substituting the above Ansätze into the Einstein equations $G_\mu^\nu = \kappa T_\mu^\nu$ leads to the following field equations

$$\begin{aligned} -\frac{D-2}{2p}e^a(a'' - p''/p) + \frac{(D-2)(D-3)}{8p}e^a(a'^2 - 2a'p'/p) \\ + \frac{(D-2)(D-7)}{8p^3}e^ap'^2 - \frac{(D-2)^2}{2ph}e^a\eta(a' - p'/p) \\ - \frac{(D-2)(D-5)}{2ph^2}e^a\eta_0^2 = -\kappa \left(U(\phi) + \frac{e^a}{2p}(2\phi'^2 - \psi'^2) + \omega_s^2 e^{-(D-3)a}\phi^2 \right) \end{aligned} \quad (11)$$

$$\begin{aligned} -\frac{(D-2)(D-3)}{8p}e^aa'^2 + \frac{(D-2)(D-3)}{8p^3}e^ap'^2 \\ + \frac{(D-2)(D-3)}{2p^2h}e^a\eta p' - \frac{(D-2)(D-3)}{2ph^2}e^a\eta_0^2 = -\kappa \left(U(\phi) - \frac{e^a}{2p}(2\phi'^2 - \psi'^2) - \omega_s^2 e^{-(D-3)a}\phi^2 \right) \end{aligned} \quad (12)$$

$$\begin{aligned} \frac{D-3}{2p^2}e^ap'' + \frac{(D-2)(D-3)}{8p}e^aa'^2 \\ + \frac{(D-3)(D-8)}{8p^3}e^ap'^2 + \frac{(D-3)^2}{2p^2h}e^a\eta p' \\ - \frac{(D-3)(D-6)}{2ph^2}e^a\eta_0^2 = -\kappa \left(U(\phi) + \frac{e^a}{2p}(2\phi'^2 - \psi'^2) - \omega_s^2 e^{-(D-3)a}\phi^2 \right) \end{aligned} \quad (13)$$

which derive from the tt , $\eta\eta$ and $\theta\theta$ components, respectively. For convenience we use the abbreviation $h = \eta^2 + \eta_0^2$.

Variation of the action with respect to the complex scalar field and to the phantom field leads to the equations

$$\phi'' + \left(\frac{D-3}{2} \frac{p'}{p} + (D-2) \frac{\eta}{h} \right) \phi' = \frac{1}{2} p e^{-a} \frac{dU}{d\phi} - \omega_s^2 p e^{-(D-2)a} \phi, \quad (14)$$

$$\left((ph)^{\frac{D-2}{2}} \frac{1}{\sqrt{p}} \psi' \right)' = 0, \quad (15)$$

where integration of the last equation leads to

$$\psi' = \sqrt{p}(ph)^{-\frac{D-2}{2}} \mathcal{D} . \quad (16)$$

Here the constant \mathcal{D} denotes the scalar charge of the phantom field. By inserting Eq. (16) into Eq. (12) the scalar charge \mathcal{D} can be expressed via

$$\mathcal{D}^2 = (ph)^{D-2} \left[\frac{(D-2)(D-3)}{4p} \left(a'^2 - p'^2/p^2 - 4\frac{\eta}{h}p'/p + 4\frac{\eta_0^2}{h^2} \right) - 2\kappa e^{-a} \left(U(\phi) - \frac{e^a}{p}\phi'^2 - \omega_s^2\phi^2 e^{-(D-3)a} \right) \right] . \quad (17)$$

We now eliminate the ϕ'^2 term and the ψ'^2 term in the Einstein equations by adding Eq. (12) to Eq. (11) and to Eq. (13). This provides us with the final set of Einstein equations

$$a'' + \frac{D-2}{h}a' - \frac{D-3}{2p}\eta a'p' = -4\kappa \frac{pe^{-a}}{(D-2)(D-3)} \left(U(\phi) - \omega_s^2(D-2)\phi^2 e^{-(D-3)a} \right) \quad (18)$$

$$p'' + \frac{(2D-5)}{h}\eta p' + \frac{D-5}{2p}p'^2 - 2\frac{(D-4)\eta_0^2}{h^2}p = -4\kappa \frac{p^2e^{-a}}{D-3} \left(U(\phi) - \omega_s^2\phi^2 e^{-(D-3)a} \right) \quad (19)$$

to be solved together with the equation for the bosonic matter, Eq. (14). Clearly, the system of equations allows for reflection symmetric solutions \mathcal{S} , i.e., solutions whose functions are either symmetric or antisymmetric under $\eta \rightarrow -\eta$.

D. Boundary Conditions

We need to solve the above set of three coupled ordinary differential equations of second order. Thus we have to impose six boundary conditions. Here we would like to impose symmetric boundary conditions, which are the same for $\eta \rightarrow \infty$ and $\eta \rightarrow -\infty$. Therefore any asymmetry in the solutions is not enforced via boundary conditions but arises spontaneously.

The boundary conditions for the boson field function $\phi(\eta)$ are then given by

$$\phi(\eta \rightarrow \pm\infty) \rightarrow 0 . \quad (20)$$

These conditions ensure, that the configurations possess finite energy.

For the metric we require asymptotic flatness in both asymptotic regions. By imposing on the metric function a the conditions

$$a(\eta \rightarrow \pm\infty) \rightarrow 0 \quad (21)$$

we also set the time scale. For the metric function p asymptotic flatness in both asymptotic regions implies

$$p(\eta \rightarrow \pm\infty) \rightarrow 1 . \quad (22)$$

E. Single and multiple throats

Here we discuss the throat properties in terms of the circumferential function $R(\eta) = \sqrt{ph}e^{-a/2}$, which represents the radius of a circle in the equatorial plane with constant coordinate η .

Let us first consider symmetric solutions \mathcal{S} . In symmetric solutions, the metric functions are symmetric under $\eta \rightarrow -\eta$. Consequently, $\eta = 0$ plays a special role. The metric functions a and p possess a vanishing derivative at $\eta = 0$, $a'(0) = p'(0) = 0$, which implies that they assume extremal values. In particular, from the circumferential radius R we can conclude, that if $R(0)$ is a minimum, a throat is located at $\eta = 0$, since $\eta = 0$ then corresponds to a minimal surface.

If $R(0)$ is a maximum, it represents only a local maximum, since $R(\eta) \rightarrow |\eta|$ asymptotically. Such a local maximum corresponds to a maximal surface and thus an equator. Clearly, between the equator and asymptotic infinity (at least) one throat should be localized in each of the regions \mathcal{M}_+ and \mathcal{M}_- . In that case double (or multiple) throat configurations are present.

Transitions between single and double throat configurations in four spacetime dimensions were observed for symmetric solutions in [12]. These transitions occur, when the second and first derivative of R vanish,

$$R''(0) = 0 \iff [(D-2)(D-3)e^a - 2p\eta_0^2\kappa U(\phi)]_{\eta=0} = 0 . \quad (23)$$

We note that for symmetric solutions the surface gravity vanishes at the throat for configurations with a single throat, since the surface gravity is given by $\frac{D-3}{2} [e^{(D-2)a/2} a' / \sqrt{p}]_{\eta_{\text{th}}}$, and $a'(0) = 0$ [12].

In contrast to symmetric solutions, where a single throat must be located at $\eta = 0$, a throat of asymmetric solutions can be located anywhere. Moreover, the transition from single to double throat solutions can happen, when the circumferential radius develops a turning point at some value of the radial coordinate $\eta_{0\text{ cr}}$, where $R'(\eta_{0\text{ cr}}) = R''(\eta_{0\text{ cr}}) = 0$. Thus for asymmetric solutions the single throat does not degenerate to form an equator and a double throat, but an equator together with a second throat appear spontaneously somewhere else in the manifold.

F. Energy conditions

When the null energy condition (NEC) is violated, also the weak and the strong energy condition are violated. It is therefore sufficient to address only the NEC, which requires

$$\Xi = T_{\mu\nu} k^\mu k^\nu \geq 0 \quad (24)$$

for all (future-pointing) null vector fields k^μ .

This condition can be expressed via the Einstein tensor by making use of the Einstein equations. For spherically symmetric solutions we then obtain the new conditions

$$-G_t^t + G_\eta^\eta \geq 0, \quad \text{and} \quad -G_t^t + G_\theta^\theta \geq 0, \quad (25)$$

both of which must be obeyed everywhere in order to respect the NEC. For the present set of solution the NEC is always violated.

G. Mass and scalar charge

In the presence of gravity, the mass of a stationary asymptotically flat solution in D dimensions can be obtained from the Komar integral [23]

$$M = -\frac{1}{16\pi G_D} \frac{D-2}{D-3} \int_{S_\infty^{D-2}} \alpha, \quad (26)$$

with $\alpha_{\mu_1 \dots \mu_{D-2}} \equiv \epsilon_{\mu_1 \dots \mu_{D-2} \rho \sigma} \nabla^\rho \xi^\sigma$ and $\xi \equiv \partial_t$. By choosing the surface S at spatial infinity in \mathcal{M}_+ (\mathcal{M}_-) we obtain the mass M_+ (M_-). Both values of the mass are encoded in the metric function g_{tt} and can easily be extracted. We obtain M_+ from

$$g_{tt} \xrightarrow{\eta \rightarrow \infty} -1 + \frac{\mu}{\eta^{D-3}}, \quad M_+ = \frac{(D-2)\Omega_{D-2}}{16\pi G} \mu, \quad (27)$$

where Ω_{D-2} is the area of the unit $D-2$ -sphere, and M_- analogously. When we use Stokes' theorem to convert the integral into a volume integral, we obtain a boundary term at some η_b , e.g.,

$$M_+ = \frac{1}{4\pi} \int_\Sigma R_{\mu\nu} n^\mu \xi^\nu dV - \frac{1}{16\pi G_D} \frac{D-2}{D-3} \int_{S_{\eta_b}^{D-2}} \alpha = \frac{1}{4\pi} \int_\Sigma R_{\mu\nu} n^\mu \xi^\nu dV + M_{\eta_b}. \quad (28)$$

Here Σ denotes an asymptotically flat spacelike hypersurface, n^μ is normal to Σ with $n_\mu n^\mu = -1$, and dV is the natural volume element on Σ [23]. Obviously, the volume integral only agrees with the mass M_+ when the surface term M_{η_b} vanishes. Since the surface term is proportional to the product of the surface gravity and the surface area, this is the case when the surface gravity vanishes.

In the symmetric case, the surface gravity always vanishes at $\eta = 0$, i.e., at the throat (or at the equator). Then the volume integral may be evaluated to give the mass. However, in the asymmetric case, the surface gravity is finite at the throat (and at the equator) [12]. In that case the surface term would have to be included to obtain the proper value of the mass from the volume integral. A simple evaluation of the volume integral from say $\eta = 0$ to $\eta = \infty$ alone would not yield the correct value for the mass in the asymmetric case. These considerations are important for evaluating the mass of the asymmetric configurations in the probe limit. Therefore we present an appropriate procedure for extracting the mass in the probe limit in Appendix A.

H. Charge or particle number

The Lagrange density is invariant under the global phase transformation

$$\Phi \rightarrow \Phi e^{i\chi} . \quad (29)$$

This leads to the conserved current

$$j^\mu = -i(\Phi^* \partial^\mu \Phi - \Phi \partial^\mu \Phi^*) , \quad j^\mu_{;\mu} = 0 . \quad (30)$$

In globally regular topologically trivial spacetimes the associated conserved charge Q is then obtained by integrating the time-component of the current over the entire space

$$\begin{aligned} Q &= - \int j^t |g|^{1/2} d\eta d\Omega_{D-2} \\ &= 2\Omega_{D-2} \omega \int_0^\infty |g|^{1/2} \frac{\phi^2}{A^2} d\eta . \end{aligned} \quad (31)$$

The global charge Q then corresponds to the particle number of the complex boson field.

In wormhole spacetimes, we have to reconsider the above definition of the charge or particle number. For symmetric configurations with a single throat it is clear that we should integrate from $\eta = 0$ to $\pm\infty$, to obtain the particle numbers Q_\pm for the regions \mathcal{M}_\pm . This should remain true, when the throat at $\eta = 0$ turns into an equator [12]. However, in the asymmetric case it is no longer obvious, where the inner integration boundary η_b should reside. If we were to retain η_b at the throat, for instance, an ambiguity would arise at the very least when the second throat and the equator emerge.

To solve this question, we reconsider the case of the mass. In the presence of gravity, the mass is obtained unambiguously and in a simple way, when we use surface integrals at plus and minus infinity. If we were to use surface integrals for the charge or particle number as well, we would also have a clean definition of the charges Q_\pm to be associated with the masses M_\pm . To achieve this feat we employ the following trick. We minimally couple the complex scalar field to a fictitious U(1) gauge field, which is not allowed to backreact on the configuration. Thus the configuration is not changed, while we can employ the Gauss law to read off the charge Q_\pm of the configurations in the asymptotically flat regions

$$Q_\pm = \int_{S_{\pm\infty}} {}^*F . \quad (32)$$

This definition then yields the same charge as the above volume integral in the symmetric case, while it leads to meaningful and unambiguous values in the asymmetric case, which can be compared with the respective masses to extract the binding energies of these configurations.

III. WORMHOLES IMMERSED IN BOSONIC MATTER

When solving the field equations subject to the given set of symmetric boundary conditions, we were in for a surprise, since the numerical procedure led to asymmetric solutions in addition to the expected symmetric solutions. Our further studies then revealed the phenomenon of spontaneous symmetry breaking in wormhole spacetimes with bosonic matter. The demonstration of this phenomenon is the main focus of this section.

While we have presented the formalism in the last section for D spacetime dimensions, we have performed the numerical calculations mainly in five dimensions, although we have also performed a study in four dimensions to convince ourselves that the analogous features are present.

In this section we first address the probe limit, where the boson field equation is solved in the background of the Ellis wormhole. Already here the spontaneous symmetry breaking is observed. Depending on the throat size, the asymmetric solutions (\mathcal{A}_\pm) can bifurcate from the symmetric solutions (\mathcal{S}) at critical values of the boson frequency. The asymmetric solutions are always energetically favoured.

Subsequently, we couple to gravity, and thus take the backreaction of the boson field into account. The phenomenon of spontaneous symmetry breaking is then retained. However, the structure of the families of solutions becomes much richer. In particular, both symmetric and asymmetric solutions exhibit transitions to double throat wormhole configurations.

Finally, we address this phenomenon in four dimensions, where contact to astrophysics can in principle be made. Here the symmetric solutions were studied in detail before, but the asymmetric ones had not been seen because the calculations had been restricted to \mathcal{M}_+ [12].

In the following we focus on the fundamental solutions. We refer to solutions as fundamental when they do not possess a radially excited boson field, i.e., when their boson field function does not have nodes. However, we have observed that the radially excited solutions exhibit the analogous pattern of symmetric and asymmetric solutions.

For the numerical calculations we have introduced the compactified radial coordinate

$$x = \text{atan} \left(\frac{\eta}{r_0} \right), \quad (33)$$

where r_0 is some constant, for which we chose $r_0 = 3$. We have then employed a collocation method for boundary-value ordinary differential equations, equipped with an adaptive mesh selection procedure [24]. We have used typical mesh sizes with $10^3 - 10^4$ points, reaching a relative accuracy of 10^{-10} for the functions. Estimates of the relative errors for the mass and the angular momentum have been of order 10^{-6} . We have employed the condition $\mathcal{D} = \text{const}$, Eq. (17), to monitor the quality of the numerical solutions. The variation of \mathcal{D} has been typically less than 10^{-9} .

For the self-interaction potential $U(\phi)$ we have chosen the parameters $\lambda = 1$, $c = 2$ and $b = 1.1$, since this allowed us to compare with previous calculations of bosonic configurations without a wormhole [25]. With these potential parameters fixed, the further parameters were the parameter κ , which includes the gravitational coupling strength, the boson frequency ω_s , and the throat parameter η_0 . In most of the results shown, we have fixed the throat parameter $\eta_0 = 3$, leaving only κ and ω_s as free parameters.

A. Probe Limit

We here demonstrate, that the phenomenon of spontaneous symmetry breaking occurs already in the probe limit. We begin by illustrating the symmetric and asymmetric solutions, and then discuss their dependence on the two parameters, the boson frequency ω_s and the throat size η_0 . We point out that depending on the throat size bifurcations occur. Subsequently, we consider the masses and particle numbers of the configurations and show that the asymmetric solutions are more strongly bound.

1. Parameters

Usually the probe limit is obtained by simply taking the coupling κ of gravity and matter to zero. However, since we want to retain the non-trivial topology, we must be careful here as to take only the coupling to the complex boson field to zero, while retaining the coupling to the phantom field. This can be achieved by an appropriate scaling of the phantom field, $\Psi \rightarrow \Psi/\sqrt{\kappa}$. The Einstein equations then yield the Ellis wormhole [13, 14], and only the boson field equation needs to be solved in the background of the Ellis wormhole.

The solutions depend on the boson frequency ω_s and on the throat parameter η_0 . In the limit $\eta_0 \rightarrow 0$, contact must be made with topologically trivial solutions, since the two asymptotically flat regions are then separated. Indeed, for $\eta_0 \rightarrow 0$, the so called non-topological soliton or Q -ball limit is reached [17, 19]. However, it can either be reached in both regions \mathcal{M}_+ and \mathcal{M}_- at the same time (symmetric case), or only in a single region (asymmetric case), as we show below.

The domain of existence of non-rotating Q -balls is restricted to a certain frequency range, $\omega_{\min} < \omega_s < \omega_{\max}$ [17, 19, 26]. The maximal frequency ω_{\max}

$$\omega_{\max}^2 = \frac{1}{2} U''(0) = \lambda b = m_b^2, \quad (34)$$

ensures that the solutions possess an exponential fall-off at spatial infinity. The minimal frequency ω_{\min} is based on an argument to allow for localized solutions [17, 19, 25, 26] and given by

$$\omega_{\min}^2 = \min_{\phi} [U(\phi)/\phi^2] = \lambda \left(b - \frac{c^2}{4} \right). \quad (35)$$

These limits are retained for the topologically non-trivial solutions, where the Minkowski background is replaced by an Ellis background [12].

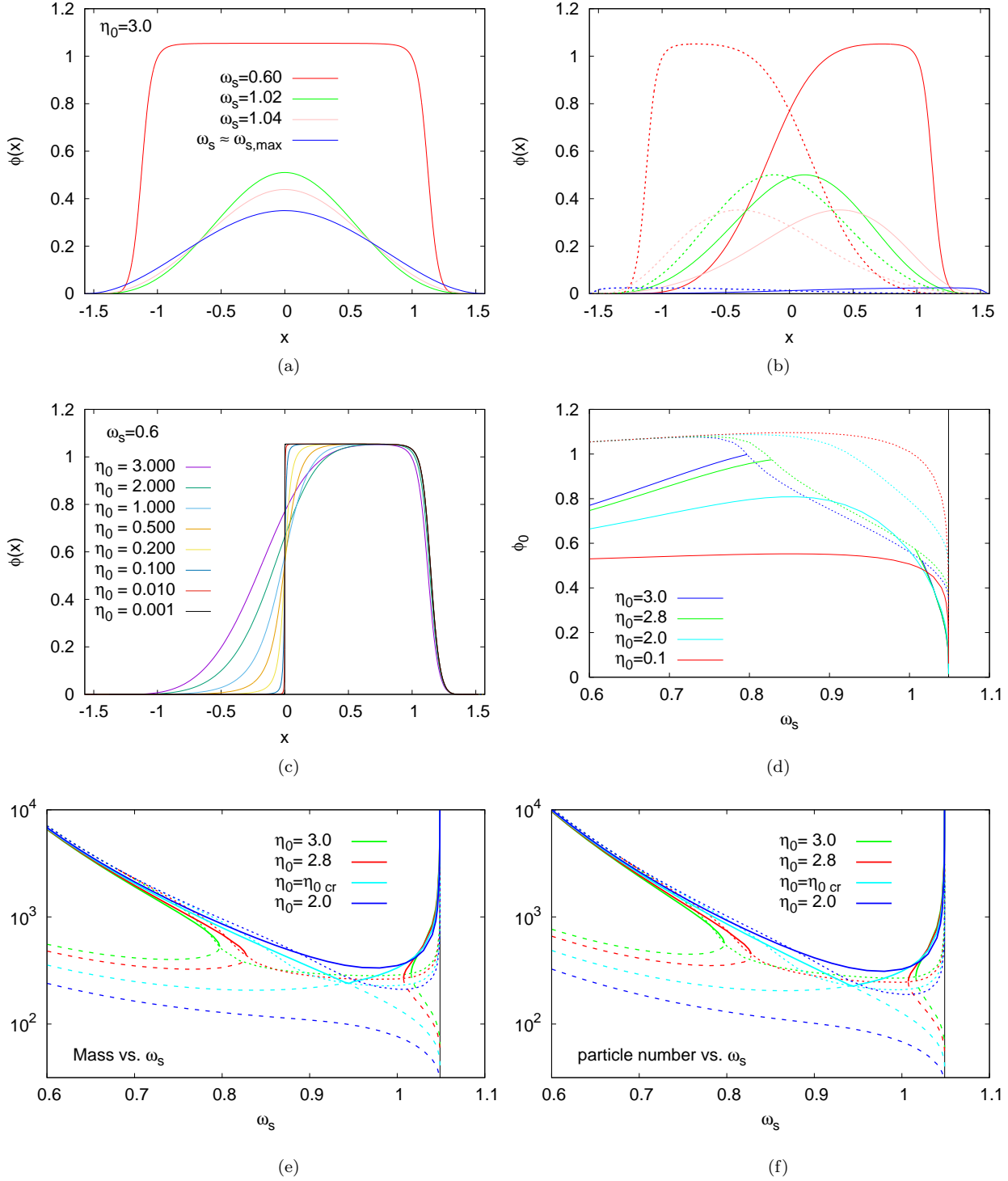


Figure 1: Probe limit: (a) The boson field function ϕ versus the compactified coordinate $x = \text{atan}(\eta/r_0)$ for a fixed throat parameter $\eta_0 = 3.0$ and decreasing values of the boson frequency ω_s . (b) Same as (a) for asymmetric solutions. (c) The boson field function ϕ versus the compactified coordinate $x = \text{atan}(\eta/r_0)$ for asymmetric solutions at a fixed boson frequency $\omega_s = 0.6$ and decreasing values of the throat parameter η_0 . (d) The value ϕ_0 of the boson field functions ϕ at the throat $\eta = 0$ versus the boson frequency ω_s for decreasing values of the throat parameter η_0 . (e) The mass M (dotted) of the symmetric solutions and the mass M_+ (solid) and M_- (dashed) of the asymmetric solutions versus the boson frequency ω_s for decreasing values of the throat parameter η_0 . (f) Same as (e) for the particle number.

2. Bifurcations

In Fig. 1 we show features of the symmetric solutions in the probe limit which are analogous to those known in four dimensions [12]. Here the boson function ϕ is symmetric with respect to the reflection $\eta \rightarrow -\eta$, as seen in Fig. 1(a), where the throat parameter η_0 is fixed while the boson frequency ω_s is varied. Note, that as the boson frequency ω_s tends to ω_{\min} , the boson function ϕ tends to a constant in an increasingly large region.

We show the boson function ϕ of the asymmetric solutions for the same parameters in Fig. 1(b). Here the reflection symmetry of the solutions is broken. However, since the field equation is reflection symmetric, all asymmetric solutions come in pairs. To distinguish the two solutions of a pair, let us include an index \pm associated with the two regions \mathcal{M}_{\pm} . The index $+$ ($-$) then indicates that more matter is localized in the region \mathcal{M}_{+} (\mathcal{M}_{-}). The functions ϕ_{-} are obtained from the functions ϕ_{+} by reflection ($\eta \rightarrow -\eta$)

$$\phi_{-}(\eta) = \phi_{+}(-\eta) , \quad (36)$$

and vice versa.

Let us now vary the throat parameter η_0 while keeping the frequency ω_s fixed. In the symmetric case, when $\eta_0 \rightarrow 0$, a *double Q-ball* solution is approached, where a *Q-ball* is localized in each of the regions \mathcal{M}_{+} and \mathcal{M}_{-} , which simply represent two disjunct Minkowski spacetimes in the limit. In contrast, in the asymmetric case for fixed frequency and $\eta_0 \rightarrow 0$, the solutions approach a *single Q-ball*, that is localized in one of the regions, either \mathcal{M}_{+} for ϕ_{+} or \mathcal{M}_{-} for ϕ_{-} , while the complementary region becomes completely empty in the limit. This is demonstrated in Fig. 1(c).

To monitor the different solutions, it is useful to keep track of the value of the boson function at the throat, ϕ_0 . This parameter then maps out the domain of existence of the symmetric and asymmetric solutions concerning the dependence on the frequency ω_s and on the throat parameter η_0 . We note, that both asymmetric solutions possess the same value of η_0 . To illustrate this dependence, we exhibit in Fig. 1(d) the value of ϕ_0 versus the boson frequency ω_s for four values of the throat parameter, $\eta_0 = 3, 2.8, 2$ and 0.1 . Recall, that for $\eta_0 = 0$ the non-trivial topology is lost.

The figure shows that a bifurcation phenomenon must take place at a critical value $\eta_{0\text{cr}}$ of the throat parameter between 2.8 and 2, that is associated with a critical value of the boson frequency ω_{cr} . For $\eta_0 \leq \eta_{0\text{cr}}$ symmetric and asymmetric solutions exist in the full interval $[\omega_{\min}, \omega_{\max}]$. At the critical value $\eta_{0\text{cr}}$, symmetric and asymmetric solutions precisely touch at the critical value ω_{cr} of the boson frequency. For $\eta_0 > \eta_{0\text{cr}}$ a frequency gap $\omega_{\text{crl}} \leq \omega_s \leq \omega_{\text{cru}}$ appears, where only symmetric solutions exist. The pairs of asymmetric solutions then bifurcate from the symmetric ones at the end points of the gap, i.e., at the upper critical frequency ω_{cru} and at the lower critical frequency ω_{crl} . The asymmetric solutions then persist for $\omega_{\min} < \omega_s < \omega_{\text{crl}}$ and $\omega_{\text{cru}} < \omega_s < \omega_{\max}$.

We note that the value of the boson field at the throat, ϕ_0 , tends to a limiting value, $\phi_0(\omega_{\min}) = 1$ in the symmetric case, when $\omega_s \rightarrow \omega_{\min}$. Here the field equation is solved by $\phi(\eta) = 1$. At first glance it may be surprising, that for small η_0 ($\eta_0 = 0.1$ in the figure) the value of ϕ_0 of the asymmetric solutions is much lower than the corresponding value of the symmetric solutions, while both approach a *Q-ball* solution in the limit $\eta_0 \rightarrow 0$. The reason for this is, that in the asymmetric case, in the limit $\eta_0 \rightarrow 0$ the ϕ_{+} field has to jump from its maximal value to zero at $\eta = 0$ (and likewise the ϕ_{-} field). Therefore for sufficiently small values of η_0 the asymmetric field assumes about half its maximal value at $\eta = 0$.

3. Mass and particle number

We now turn to the global charges of these solutions, beginning with the mass. In Fig. 1(e) we exhibit the mass versus the boson frequency ω_s for several values of the throat parameter η_0 , including the critical value $\eta_{0\text{cr}}$. The mass M of the symmetric solutions is shown by dotted curves, the mass M_{+} of the solutions with more matter localized in \mathcal{M}_{+} is represented by solid curves, and the mass M_{-} of the solutions with more matter localized in \mathcal{M}_{-} by dashed curves. We recall that these masses refer to the asymptotic behaviour in \mathcal{M}_{+} . Because of the symmetry of the pair of asymmetric solutions, the values of their masses would be interchanged when read off in \mathcal{M}_{-} .

The figure nicely illustrates the bifurcation phenomenon seen already in Fig. 1(d). Below $\eta_{0\text{cr}}$ there are three distinct curves for M , M_{+} and M_{-} in the full frequency range. At $\eta_{0\text{cr}}$ the three curves touch at the critical boson frequency ω_{cr} , while beyond $\eta_{0\text{cr}}$ there are two bifurcation points of the boson frequency, ω_{crl} and ω_{cru} , where the pairs of asymmetric solutions bifurcate from the symmetric ones. Away from such bifurcation points, the mass M_{-} is much smaller than the other masses.

The particle number Q is exhibited in Fig. 1(f). Employing the same style for the respective Q curves as for the M curves, we see, that the dependence of the particle number on the throat size and on the boson frequency is completely analogous to the dependence of the mass.

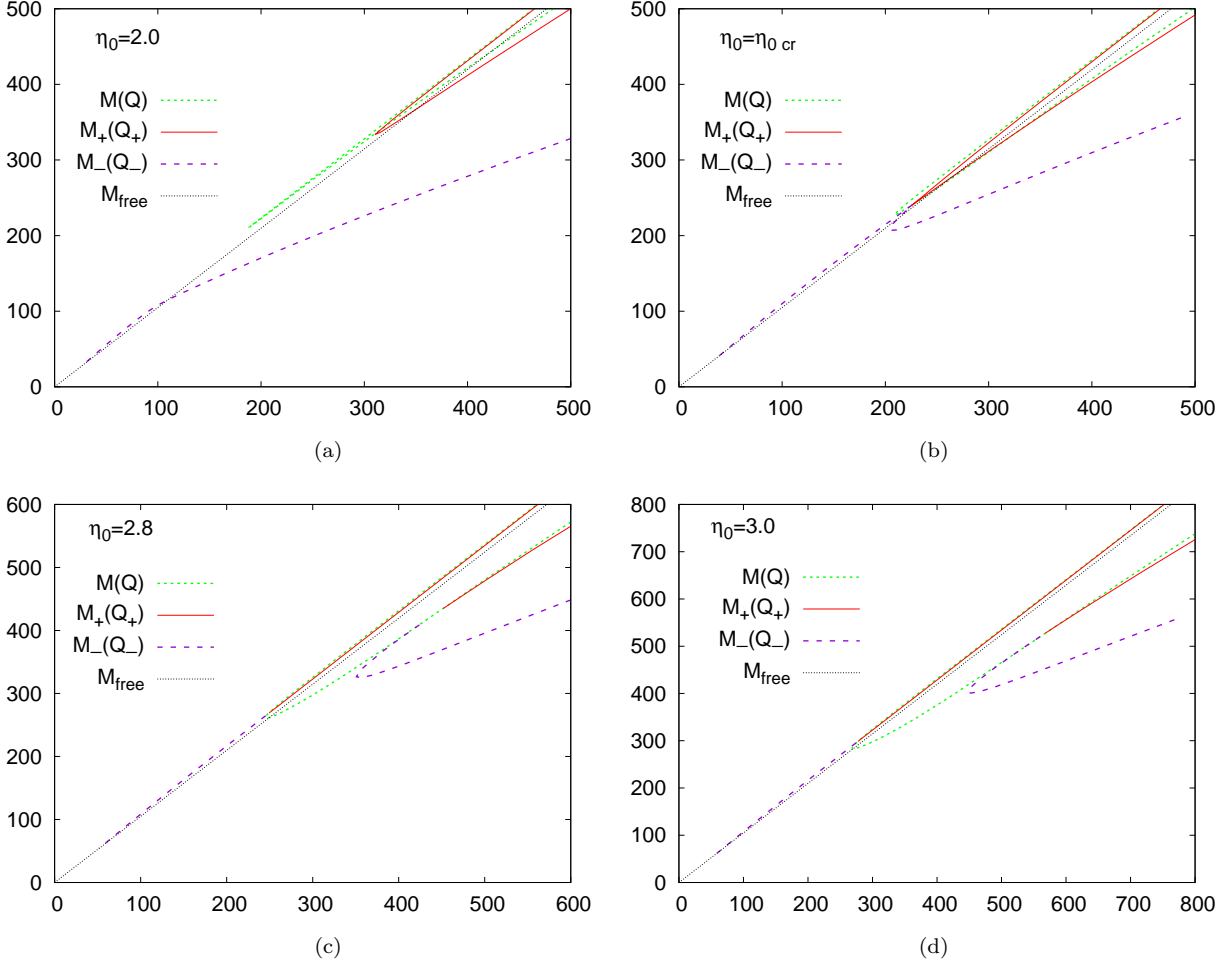


Figure 2: Probe limit: (a) The mass M (dotted) of the symmetric solutions and the mass M_+ (solid) and M_- (dashed) of the asymmetric solutions versus the respective particle number Q , Q_+ , and Q_- for throat parameter $\eta_0 = 2$ in \mathcal{M}_+ . (b) Same as (a) for the critical value of η_0 . (c) Same as (a) for $\eta_0 = 2.8$. (d) Same as (a) for $\eta_0 = 3$. The mass of Q free bosons is also shown (thin-dotted).

Let us now demonstrate that due to spontaneous symmetry breaking the asymmetric solutions are energetically favourable. To this end we address the question, in which of the solutions the bosons are most strongly bound. The mass of Q free bosons is given by

$$M_{\text{free}} = m_b Q. \quad (37)$$

The binding energy of solutions with Q (Q_{\pm}) particles can then be extracted from the difference of their mass M (M_{\pm}) and the mass of Q (Q_{\pm}) free bosons.

When considering $M(Q)$, $M_+(Q_+)$ and $M_-(Q_-)$ one obtains cusp-like structures, as illustrated in Fig. 2, where the masses M , M_+ and M_- are shown versus their respective particle numbers for the same set of throat parameters as in Fig. 1(e). Also shown in the figure is the mass of Q free bosons.

The symmetric solutions always form a cusp at some minimal value of the mass and the particle number. From this cusp two branches emerge, where the lower branch soon becomes bound, i.e., $M < M_{\text{free}}$, whereas the upper branch remains unbound.

The branch structure of the asymmetric solutions depends on the throat parameter and the bifurcation phenomenon. For $\eta_0 < \eta_{0\text{ cr}}$, the mass M_+ of the asymmetric solutions possesses an analogous cusp structure to the symmetric one. However beyond $\eta_{0\text{ cr}}$ the respective upper and lower branch are no longer connected. Then one clearly notices the bifurcations at $\omega_{\text{cr}l}$ and $\omega_{\text{cr}u}$, where in each case two asymmetric branches bifurcate from a symmetric branch.

In the vicinity of each bifurcation point, both of the emerging asymmetric solutions possess a lower mass for the same particle number than the respective symmetric solutions. Thus they are energetically favoured, and they remain

energetically favoured, also far from their respective bifurcation points. As expected, the spontaneous symmetry breaking leads to energetically more favourable solutions.

B. Gravitating Solutions

We now consider the backreaction of the boson field on the wormhole. This means that the full set of coupled nonlinear Einstein-matter equations is solved. Then in addition to the throat parameter η_0 and the boson frequency ω_s the coupling constant to gravity, which is contained in the parameter κ , enters as another continuous parameter. To reduce the resulting amount of data, we here fix the throat parameter to $\eta_0 = 3$, a value above the critical value. This will allow us to see, how the coupling to gravity affects the symmetry breaking. Again we will focus on solutions in five dimensions.

Fixing the values of κ and η_0 (as well as the number of dimensions), we then obtain families of gravitating solutions, formed again for both symmetric (\mathcal{S}) and asymmetric (\mathcal{A}_\pm) solutions, which depend on the boson frequency ω_s . However, unlike the case of the probe limit, the dependence of these families of solutions on ω_s can become very involved in the presence of gravity. Moreover, the wormhole geometry changes at certain frequencies from single throat to double throat configurations.

1. Symmetry Breaking

Let us begin by considering the spontaneous symmetry breaking in the presence of gravity. In Fig. 3 we illustrate the dependence of the families of gravitating solutions on the coupling constant κ , keeping the throat parameter η_0 fixed. We start with a rather small value of the coupling constant κ . In particular, we exhibit in Fig. 3(a) the masses versus the boson frequency for $\kappa = 0.001$ and in Fig. 3(b) the respective particle numbers. We then increase the coupling to $\kappa = 0.01$ and $\kappa = 0.1$, where the respective masses are shown in Fig. 3(c) and Fig. 3(d).

First of all we observe that the symmetry breaking persists when gravity is coupled. Thus besides the family of symmetric solutions also the families of asymmetric solutions are retained. The bifurcation frequencies ω_{crl} and ω_{cru} , where the pairs of asymmetric solutions bifurcate from the symmetric solutions, vary only slightly with increasing κ . As in the probe limit, in between these two bifurcation frequencies only symmetric solutions exist.

Clearly, in a large range of boson frequencies, the structure of the gravitating solutions follows the structure of the solutions in the probe limit. In fact, as seen in Fig. 3(a) and 3(b), the solutions possess very similar global charges there. However, for very large frequencies and for small frequencies significant deviations occur. In particular, a third bifurcation frequency $\omega_{\text{crg}} < \omega_{\text{crl}}$ arises, where the asymmetric solutions merge again with the symmetric ones. This bifurcation is highlighted in the inset in Fig. 3(c). This new gravity induced bifurcation at boson frequency ω_{crg} marks the endpoint of the asymmetric branches.

Let us now go into more detail. For small κ a very prominent new feature is the spiral structure at large values of the mass and particle number. This structure is present for both symmetric and asymmetric solutions, and resembles at first glance very much the spiral structure of boson stars. Indeed, the branches of solutions of topologically non-trivial solutions follow closely the branch of boson stars up into the spiral. However, unlike boson stars, the present spirals unwind again as seen in the inset in Fig. 3(a).

From a physical point of view, we consider that the most relevant branches of solutions are those that start from the minimal value of M (\mathcal{S}) or from the bifurcation frequency ω_{crl} (\mathcal{A}_\pm) and continue to smaller frequencies until (for the \mathcal{S} and \mathcal{A}_+ solutions) the mass and the particle number reach a maximum. At this maximum a change of the stability of the solutions is expected to occur, in analogy to boson stars. Namely the solutions should acquire an (additional) unstable mode. By symmetry the \mathcal{A}_- would of course also change stability at the same frequency.

As the spirals unwind, the symmetric and asymmetric configurations change their geometry and evolve a double throat structure, as indicated by the change of colour (blue) in the figures. This will be discussed in more detail below. Here we note that the families of solutions then bend backwards and form descending branches with respect to their global charges, which follow closely those of the physically more relevant ascending branches. When the descent is slowed down, the solutions approach the third bifurcation frequency ω_{crg} , where the asymmetric solutions merge again with the symmetric ones. Only this symmetric branch then continues to smaller frequencies ω_s .

We remark, that for frequencies very close to the maximum frequency also significant deviations from the probe limit arise, not discerned in Fig. 3(a). However, the inset of Fig. 3(b), where the particle number is shown, reveals that some interesting further branch structure is present close to ω_{max} for these small values of the coupling κ . In general, the dependence of the particle number on the frequency follows very closely the dependence of the mass. We therefore exhibit the particle number only for a single coupling constant κ . The only noticeable exception to this

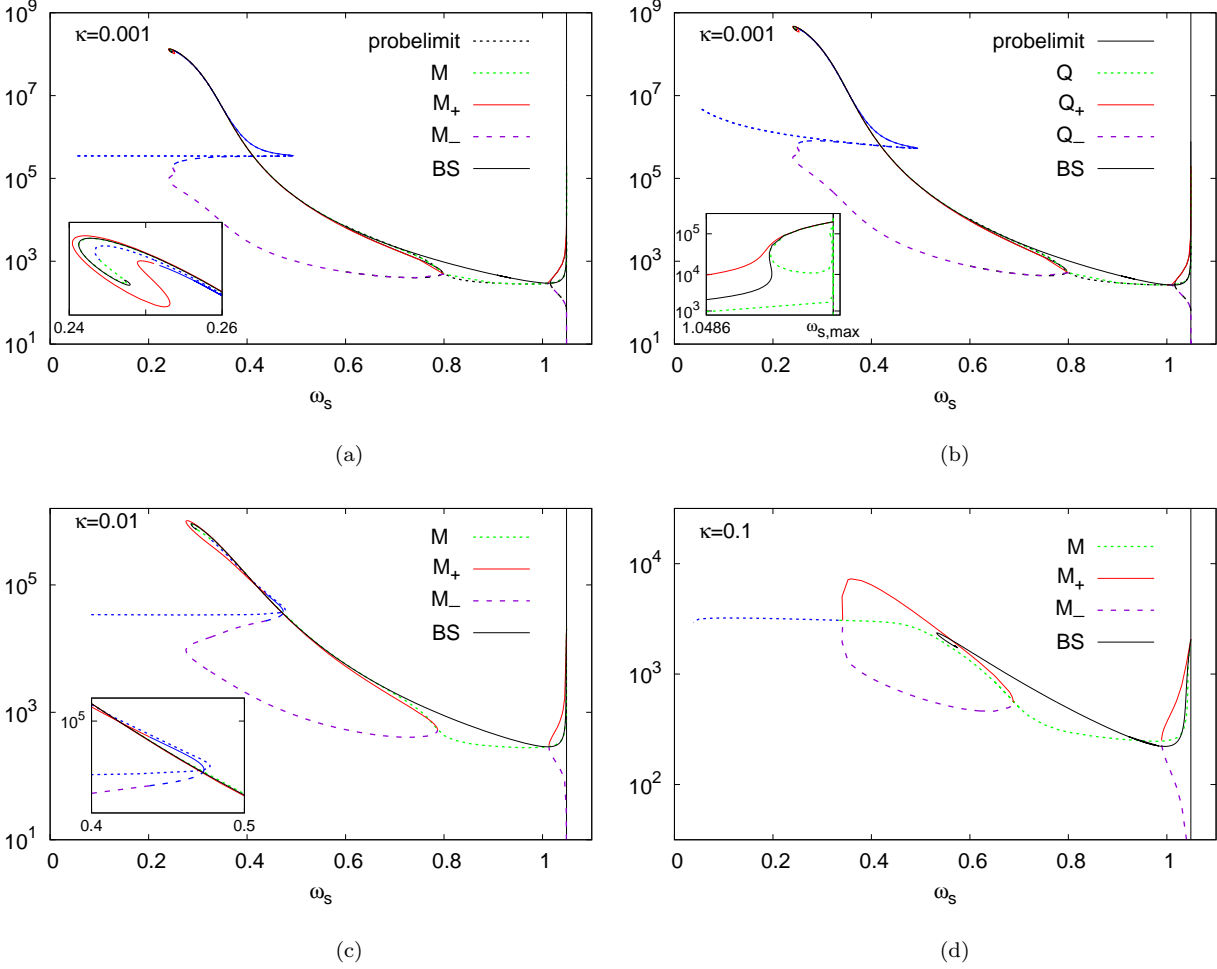


Figure 3: Properties of gravitating solutions versus the boson frequency ω_s ($\eta_0 = 3$). (a) $\kappa = 0.001$: The mass M of the symmetric solutions (dotted green), and the masses M_+ (solid red) and M_- (dashed lilac) of the asymmetric solutions. For double throat configurations the colour is changed to blue. (b) Same as (a) for the particle number Q . (c) Same as (a) for $\kappa = 0.01$ (d) Same as (a) for $\kappa = 0.1$. Also indicated are the masses of the respective boson stars (solid black). The mass and particle number in the probe limit are shown in (a) and (b) for $\omega_s > 0.6$ (dotted black). The thin vertical lines indicate ω_{\max} .

behaviour occurs for $\omega_s < \omega_{\text{crg}}$ on the last part of the branch of symmetric solutions (independent of κ). Here the particle number increases, whereas the mass remains almost constant.

To address the κ -dependence of the solutions, we exhibit in Fig. 3 also the masses for $\kappa = 0.01$ (c) and $\kappa = 0.1$ (d). For $\kappa = 0.01$ the spiral structure is still present at a rudimentary level. Thus the general structure of the solutions is analogous to the one of the lower couplings. When κ is further increasing, however, the spiral behaviour disappears, and the branch structure of the families of solutions radically simplifies. This also holds for frequencies close to the maximal frequency. The mass and the particle number of a given family are then simple functions of the boson frequency ω_s , since the backbendings present for lower κ disappear, as illustrated for $\kappa = 0.1$.

We have focussed our discussion here on the fundamental solutions. But we would like to remark, that the families of first excitations follow the pattern of the fundamental solutions.

Let us now consider the $M(Q)$ dependence in order to demonstrate that the asymmetric solutions are energetically favoured also in the presence of gravity. To this end we exhibit in Fig. 4 the masses M , M_+ and M_- versus the respective particle numbers Q , Q_+ and Q_- for several values of the coupling constant κ . Also shown is the mass of Q free bosons for comparison. Fig. 4(a) presents the masses versus the particle numbers for the small coupling $\kappa = 0.001$, zooming into the small mass region, where the bifurcations at ω_{crl} and ω_{cru} are clearly visible.

As expected, for these small masses and particle numbers there is very little difference to the probe limit, except for an overall scaling parameter (compare Fig. 2(d)). Thus the asymmetric solutions are clearly energetically favoured, when they bifurcate from the symmetric ones.

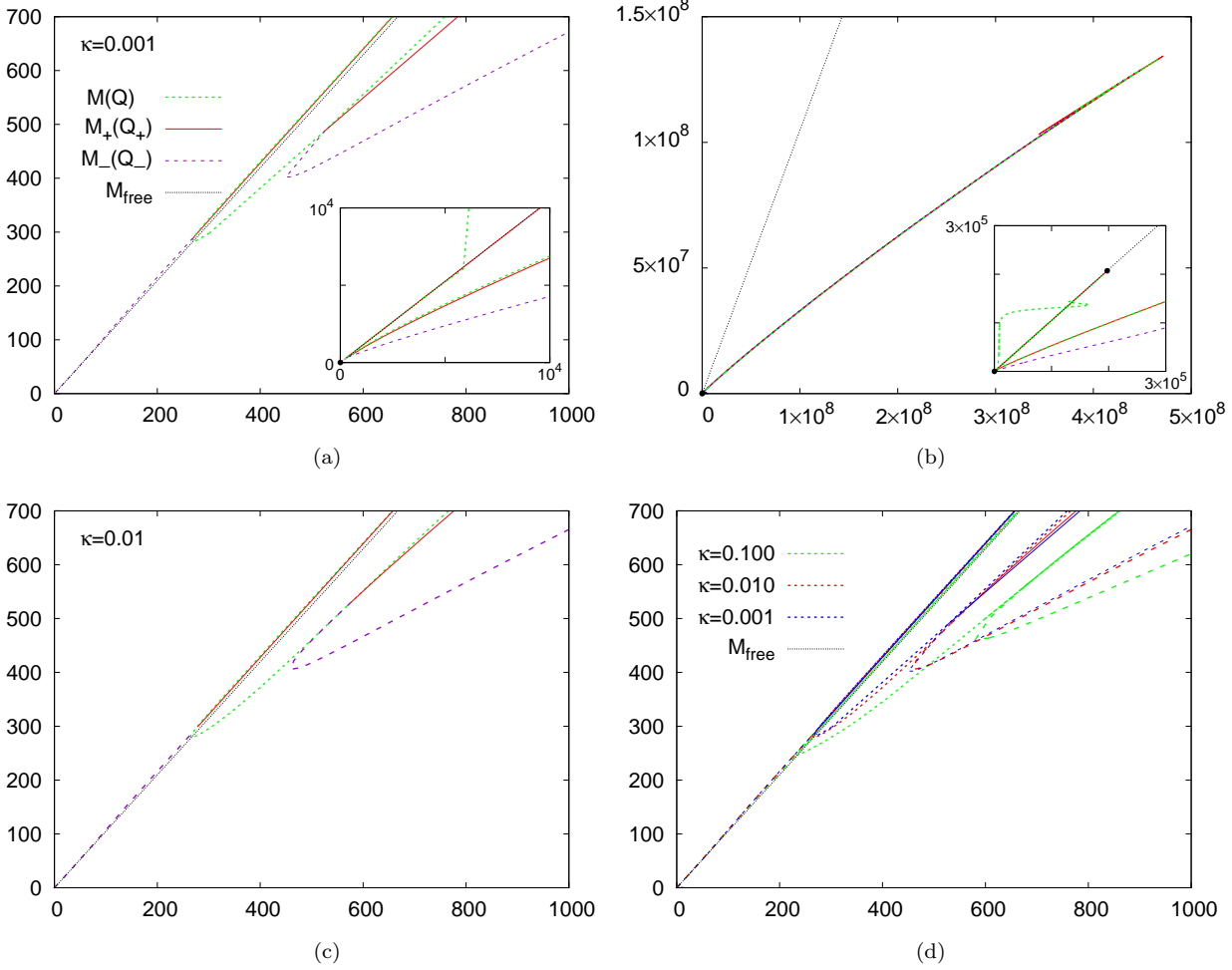


Figure 4: Gravitating solutions for throat parameter $\eta_0 = 3$: (a) The mass M (dotted) of the symmetric solutions and the mass M_+ (solid) and M_- (dashed) of the asymmetric solutions versus the particle number Q , Q_+ , and Q_- , respectively, for coupling $\kappa = 0.001$ in the vicinity of the bifurcations ω_{crI} and ω_{crU} . (b) Same as (a) for the full range of masses. (c) Same as (a) for $\kappa = 0.01$. (d) Same as (a) for $\kappa = 0.1$, $\kappa = 0.01$ and $\kappa = 0.001$. Note the different colour coding here. In all figures the mass of Q free bosons is shown for comparison (thin black lines). The black dot in the inset in (b) denotes the endpoint at ω_{max} .

But the inset in Fig. 4(a) shows already a part of the additional structure of the symmetric solutions close to ω_{max} . The full structure of the solutions is shown in Fig. 4(b). Interestingly, the various branches of $M(Q)$ for the symmetric and asymmetric solutions are all very close to each other on such a large scale. The large mass solutions are all strongly bound as a comparison with the free case shows. The spiral structure gives rise to a number of cusps at large masses. However, since the masses and particle numbers are all very close, we cannot discern these cusps in the figure. (Here only a schematic plot would clearly exhibit the cusp structure.)

The inset in Fig. 4(a) shows already a part of the additional structure of the symmetric solutions close to ω_{max} . The black dot in the inset in Fig. 4(b) denotes the endpoint reached by the configurations at ω_{max} . This endpoint is universal, i.e., independent of the coupling κ . It also represents a fourth bifurcation point, since the asymmetric and symmetric solutions merge again at ω_{max} .

In Fig. 4(c) we demonstrate the $M(Q)$ dependence for $\kappa = 0.01$ in the small mass region. The figure shows that the asymmetric solutions remain energetically preferred, when the coupling is increased. We observe that the figures 4(a)-(c) look very much the same. This is demonstrated in 4(d) for all values of κ considered. Indeed, there is only a slight dependence on the coupling κ apart from the overall scaling factor. Clearly, the asymmetric solutions remain preferred.

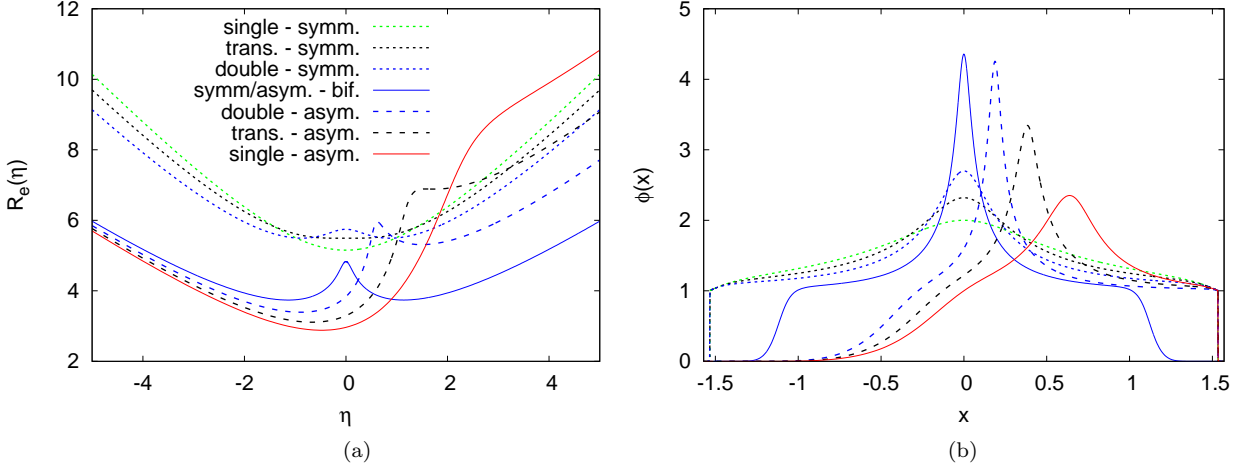


Figure 5: Throat structure of non-rotating gravitating solutions ($\eta_0 = 3$). (a) The circumferential radius function $R(\eta)$ ($R_+(\eta)$) versus the compactified coordinate $x = \text{atan } \eta$ for a sequence of symmetric and asymmetric solutions with various values of the frequency ω_s and coupling constant $\kappa = 0.001$. (b) Same as (a) for the boson function $\phi(\eta)$ ($\phi_+(\eta)$).

2. Wormhole Geometries

Next we address the geometry of the solutions. In particular, we are interested in the transition from single throat configurations to configurations featuring a double throat with an equator in between the two throats. To illustrate this transition, we exhibit in Fig. 5(a) the circumferential radius function $R(\eta)$ ($R_+(\eta)$) for a sequence of symmetric (asymmetric) solutions, where the coupling constant has the value $\kappa = 0.001$. Here a minimum of $R(\eta)$ ($R_+(\eta)$) corresponds to a throat, and a maximum to an equator. The transition from a single throat to a double throat configuration occurs at an inflection point.

For the symmetric solutions we display in Fig. 5(a) a single throat solution for the frequency $\omega_s = 0.2456$, the transition from a single to a double throat configuration happening at $\omega_{\text{tr}} = 0.2439$, and a double throat solution with $\omega_s = 0.2435$. The asymmetric double throat solutions bifurcate at $\omega_{\text{crg}} = 0.4942$ from the symmetric ones. For the asymmetric solutions the respective solutions have frequencies $\omega_s = 0.2724$ (double throat), $\omega_{\text{tr}} = 0.2518$ and $\omega_s = 0.2494$ (single throat). As seen in the figure, in the asymmetric case the equator and the second throat emerge asymmetrically. Thus the single throat does not degenerate at the transition frequency, where the equator and the second throat arise. Instead, an inflection point arises in the other part of the manifold, which splits into a maximum and a minimum as the frequency ω_s is increased.

In addition we display in the figure the double throat solution at the third bifurcation frequency ω_{crg} , where the symmetric and asymmetric solutions merge again. To get an idea of the matter distributions associated with these configurations and, in particular, with the transitions, we exhibit in Fig. 5(b) the boson function $\phi(\eta)$ ($\phi_+(\eta)$) for the same set of solutions. We observe that the value of ϕ_0 increases from the single to the double throat solutions. In particular, we note that the inflection point arises close to the peak of $\phi_+(\eta)$ in the part of the manifold, where most of the matter resides.

To see the evolution of the throats of the symmetric and asymmetric configurations, we exhibit the dependence of their locations in Fig. 6, beginning in Fig. 6(a) with the coupling $\kappa = 0.001$. Clearly, for symmetric solutions their single throat (solid green) is localized at $\eta_t = 0$, while beyond their transition frequency ω_{tr} their equator resides at $\eta_e = 0$ (dotted black). At the transition frequency the double throats (solid and dashed black) emerge and stride away.

For the larger frequencies the single throat of the asymmetric solutions is located close to $\eta_t = 0$. However, for the solutions \mathcal{A}_+ it is then shifted into the region \mathcal{M}_- (and vice versa) (solid dark red), because of the backreaction of the matter on the geometry. At the transition frequency this throat continues to stride away (solid dark blue), while a cusp arises in the region \mathcal{M}_+ , formed by the second throat (dashed dark blue) and the equator (dotted dark blue). Note that the light colours (red and blue) in the figure represent the second asymmetric solution. At the bifurcation ω_{crg} , the asymmetric throats merge with the symmetric throats, and the asymmetric equator with the symmetric one.

In Fig. 6(b) we zoom into the region close to ω_{max} . Here only single throat solutions exist. The inset demonstrates the merging of the solutions at ω_{max} . We note that in this region the asymmetric throats first stride far away from $\eta = 0$, before they return towards the limiting symmetric solution.

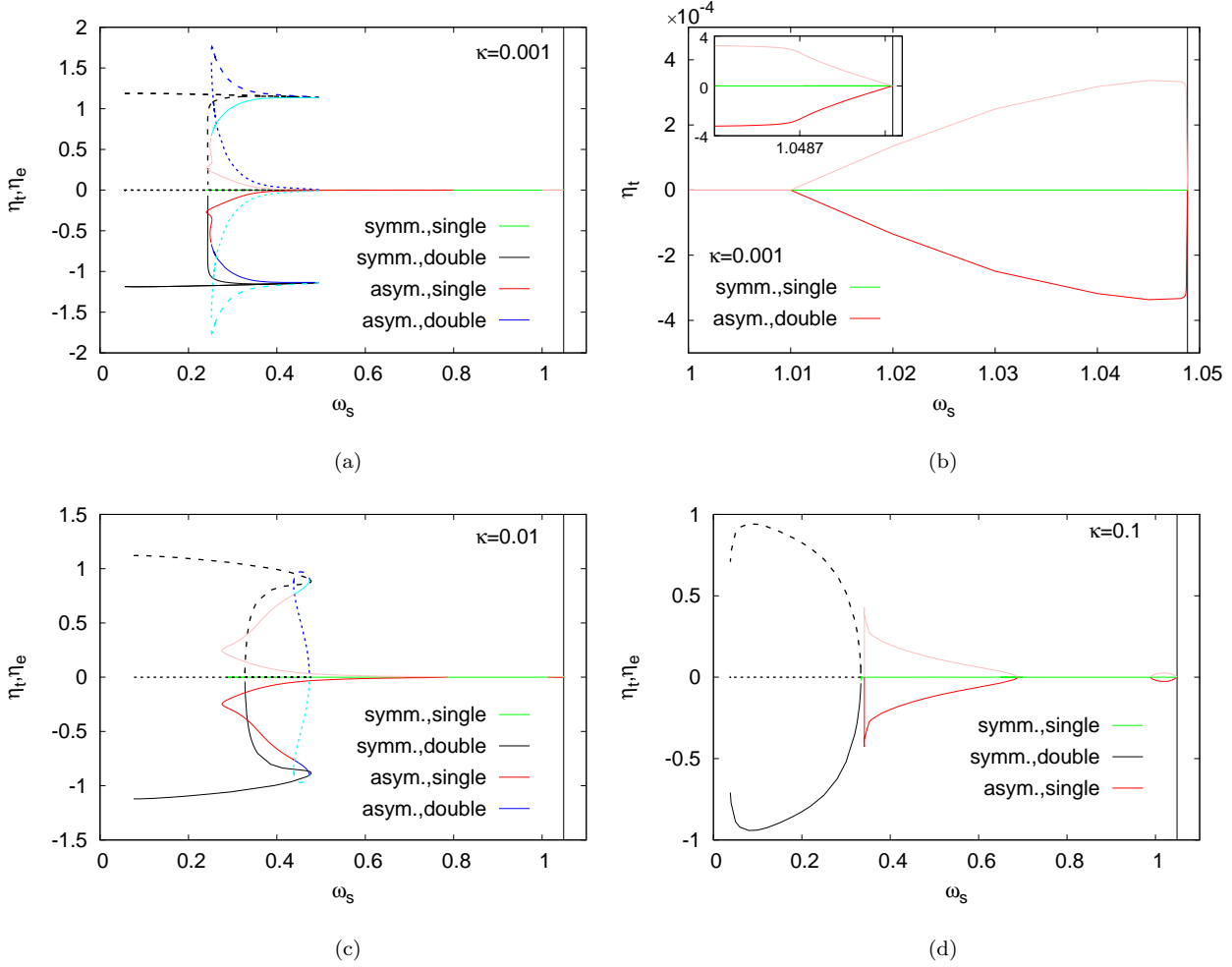


Figure 6: Gravitating solutions for throat parameter $\eta_0 = 3$: (a) Location of the single throat respectively the double throat and the equator versus the frequency ω_s for coupling $\kappa = 0.001$. (b) Zoom of (a) for frequencies close to ω_{\max} . (c) Same as (a) for $\kappa = 0.01$. (d) Same as (a) for $\kappa = 0.1$.

As κ is increased, the frequency range of the asymmetric double throat solutions decreases, as seen in Fig. 6(c) for $\kappa = 0.01$, where the same colour coding is used. For large κ only symmetric double throat solutions are left, as illustrated in Fig. 6(d) for $\kappa = 0.1$. These symmetric double throat solutions persist at small frequencies for any value of the coupling κ .

Let us now visualize the geometry of the wormholes by considering embeddings of spatial hypersurfaces. We display an isometric embedding of one of the equatorial planes ($\theta = 0, \pi/2$) for several wormhole spacetimes in Fig. 7. For the embedding we employ the parametric representation

$$\rho(\eta) = R(\eta), \quad z(\eta) = \int_0^\eta \sqrt{1 - R'^2} \, d\eta'. \quad (38)$$

Below the transition value ω_{tr} the solutions possess a single throat, as seen in Fig. 7 for a symmetric (a) and an asymmetric (b) solution. For the parameters chosen in the figure, $\kappa = 0.001$ and $\omega_s = 0.3$, the single throat of the asymmetric solution has been pushed into \mathcal{M}_- . The respective double throat solutions are shown in Fig. 7(c) and (d). For the symmetric ones the equator resides at the centre (c), while for the asymmetric ones it is located off the centre (d). We did not find solutions with more than two throats.

Finally, we would like to briefly comment on the limiting solution for small boson frequencies. As discussed above, for small ω_s only symmetric solutions persist. When ω_s is decreased here, the boson function becomes steeper and steeper in the vicinity of the equator, with its peak strongly increasing. At the same time the circumferential function $R(\eta)$ peaks more and more strongly at the equator. This behaviour indicates that a singular limiting solution is reached, whose Kretschmann scalar diverges in the limit. This agrees with the previous four-dimensional study [12].

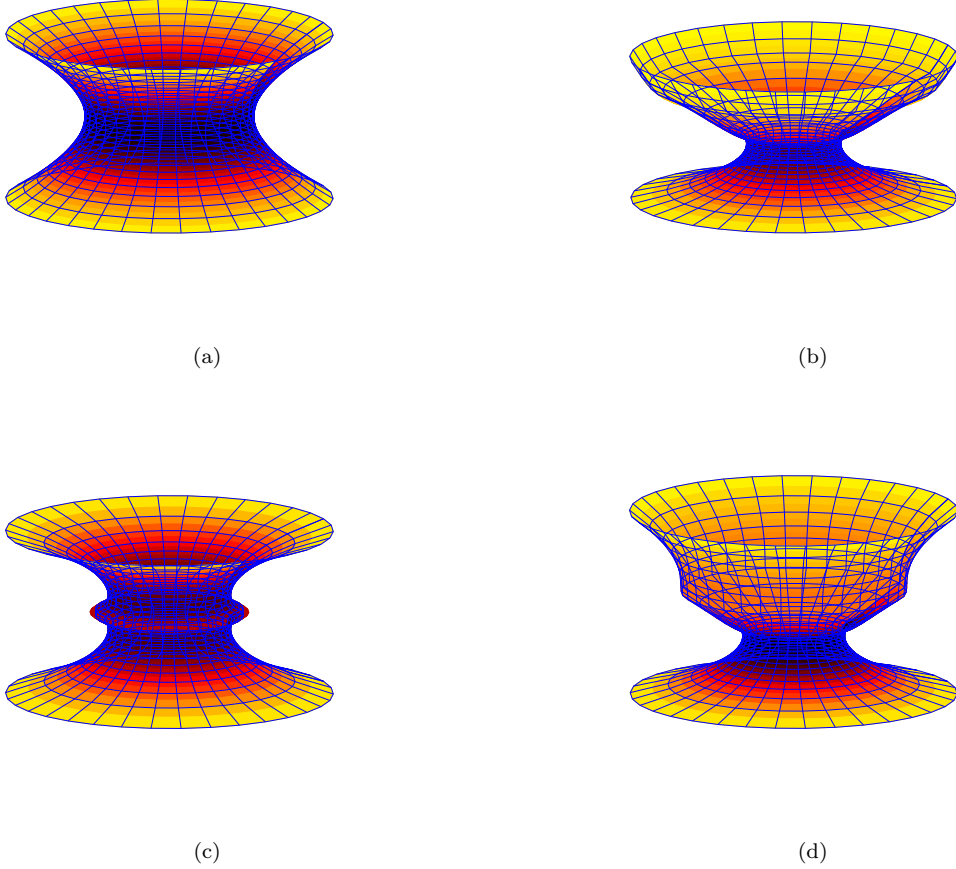


Figure 7: Throat geometry: Three dimensional view of the isometric embedding. (a) Symmetric solution with a single throat for $\kappa = 0.001$, $\omega = 0.3$. (b) Asymmetric solution with a single throat for $\kappa = 0.001$, $\omega = 0.3$. (c) Symmetric solution with a double throat for $\kappa = 0.001$, $\omega = 0.3$. (d) Asymmetric solution with a double throat for $\kappa = 0.001$, $\omega = 0.3$.

C. Spontaneous Symmetry Breaking in Four Spacetime Dimensions

After having discussed in detail the properties of the wormholes within bosonic matter in five spacetime dimensions, we now briefly demonstrate that the features observed in five dimensions also hold in four dimensions. We recall, that in [12] only the symmetric solutions have been studied, which we now complement with the asymmetric ones.

First of all we note, that the analogous bifurcation phenomenon takes place in four dimensions. In the probe limit, below a critical value $\eta_{0\text{cr}}$ of the throat parameter, symmetric and asymmetric solutions exist in the full frequency interval $[\omega_{\min}, \omega_{\max}]$. Above this critical value a frequency gap $\omega_{\text{crl}} \leq \omega_s \leq \omega_{\text{cru}}$ appears, where only symmetric solutions are found. At the end points of the gap, i.e., at ω_{cru} and ω_{crl} , pairs of asymmetric solutions bifurcate from the symmetric ones, and persist for $\omega_{\min} < \omega_s < \omega_{\text{crl}}$ and $\omega_{\text{cru}} < \omega_s < \omega_{\max}$.

As in five dimensions, this bifurcation phenomenon is retained in the presence of gravity. Depending on the coupling constant κ , the branch structure of the solutions changes considerably, as already demonstrated in [12] for the symmetric solutions (and for a smaller throat parameter). The asymmetric solutions follow this general pattern. Thus with increasing κ the branch structure simplifies, and the spiral part disappears, analogously to the five-dimensional solutions. We here demonstrate the branch structure for the value of $\kappa = 0.01$.

Fig. 8(a) exhibits the masses M (dotted green), M_+ (solid red) and M_- (dashed lilac), where the bifurcations are clearly visible. The asymmetric solutions emerge from the symmetric ones at the bifurcations ω_{crl} and ω_{cru} , and merge again at ω_{crg} (as well as at ω_{\max}). Also, the transition from single to double throat solutions is observed (blue) analogously to the five-dimensional case.

The respective particle numbers are shown in Fig. 8(b). Again, they follow the behaviour of the masses rather closely, except for very small frequencies ω_s , where only the symmetric solutions are retained. In Fig. 8(c), the masses M , M_+ and M_- are shown versus the respective particle numbers Q , Q_+ and Q_- in the bifurcation region (i.e., for

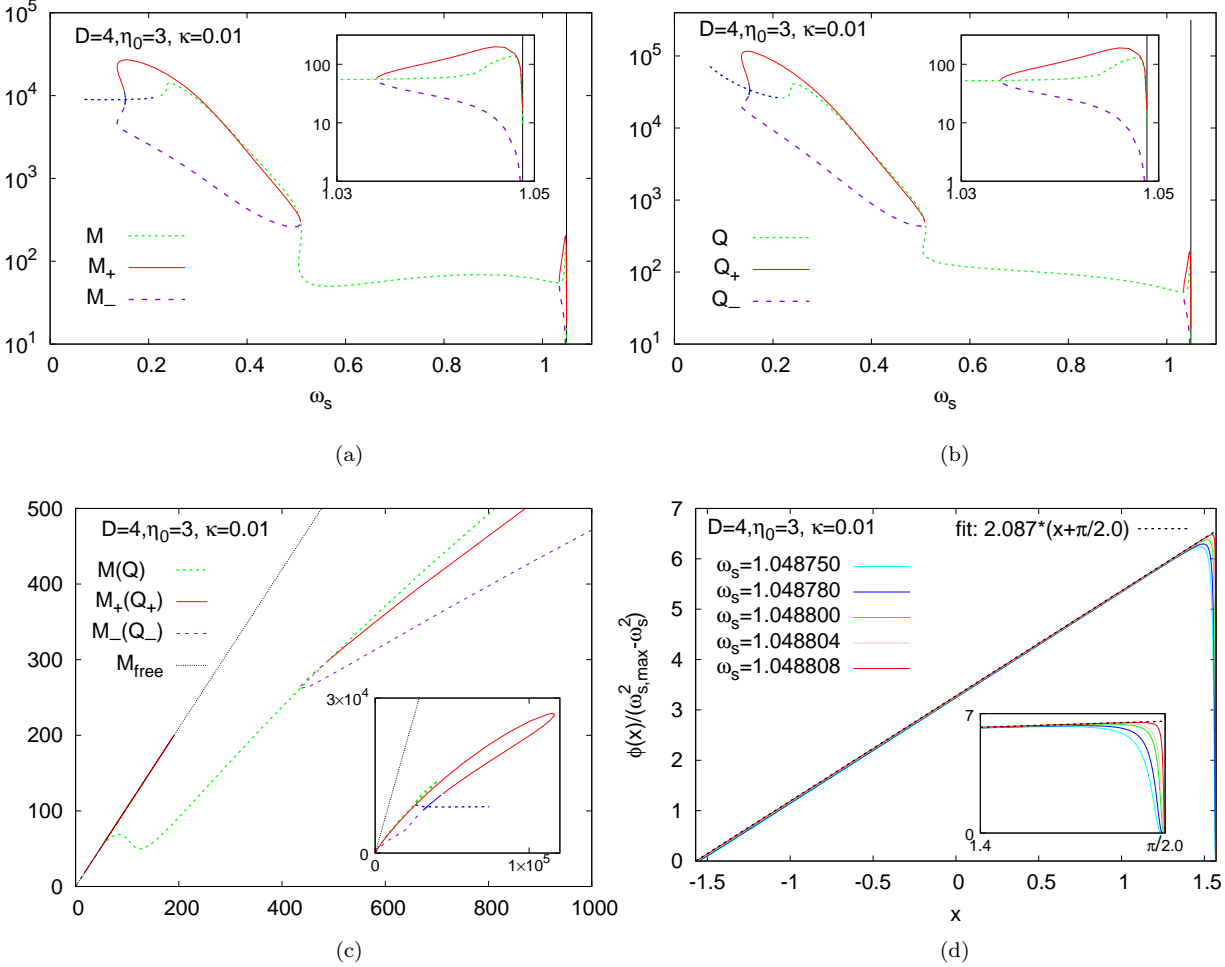


Figure 8: Properties of gravitating solutions versus the boson frequency ω ($\eta_0 = 3$) in four dimensions. (a) $\kappa = 0.01$: The mass M of the symmetric solutions (dotted green), and the masses M_+ (solid red) and M_- (dashed lilac) of the asymmetric solutions. For double throat configurations the colour is changed to blue. (b) Same as (a) for the particle number Q . (c) The mass M (dotted) of the symmetric solutions and the mass M_+ (solid) and M_- (dashed) of the asymmetric solutions versus the particle number Q , Q_+ , and Q_- , respectively, for coupling $\kappa = 0.01$ in the vicinity of the bifurcations ω_{cr1} and ω_{cr2} . (d) The limit $\omega_s \rightarrow \omega_{\text{max}}$ is demonstrated for the boson field function ϕ .

the larger values of the frequency ω_s), together with the mass of Q free bosons. The inset in Fig. 8(c) exhibits the solutions in their full range of existence.

As expected, the asymmetric solutions are energetically favoured over the symmetric ones. Thus the spontaneous symmetry breaking leads to more strongly bound systems. We conjecture, that the analogous phenomenon is present also in more than five dimensions. Let us remark, that in four dimensions we are in principle entitled to also address astrophysical aspects of these solutions. Then one may view them as boson stars with a nontrivial topology. However, this aspect will be addressed elsewhere.

As a final point, we remark that in four dimensions the limiting solution for $\omega_s \rightarrow \omega_{\text{max}}$ becomes rather trivial in the sense, that the boson field tends to zero and the solution has vanishing mass. This is different for five dimensions, where the limiting solution has a non-trivial boson field and a finite mass. We exhibit in Fig. 8(d) a sequence of solutions close to ω_{max} , which demonstrates the way the boson field approaches its limit.

IV. CONCLUSIONS AND OUTLOOK

In this paper we have considered Ellis wormholes in the presence of a complex bosonic matter field and encountered the phenomenon of spontaneous symmetry breaking of the solutions. Starting with the probe limit, we have seen

that besides the symmetric configurations there are also asymmetric configurations present. The latter always come in pairs, and are related to each other by a reflection with respect to the radial coordinate $\eta = 0$.

For a small throat size, the symmetric and asymmetric solutions are present in the full frequency interval $[\omega_{\min}, \omega_{\max}]$, whereas for a larger throat size this is only true for the symmetric solutions. There the asymmetric solutions branch off the symmetric ones at critical values of the frequency, ω_{crI} and ω_{crU} . At the critical value of the throat size $\eta_{0\text{ cr}}$ these critical frequency values coincide.

Both symmetric and asymmetric solutions satisfy the same set of boundary conditions. In that sense the asymmetric solutions appear spontaneously, without any external trigger. The reason for their appearance is that the asymmetric solutions are energetically favourable, as we have shown.

To this end, we have analyzed the masses and particle numbers of the solutions. For the symmetric solutions one finds the same mass M in both asymptotic regions, and likewise the same particle number. For the asymmetric solutions this is different. Here the solution with most of its mass located in the region \mathcal{M}_+ possesses the mass M_+ in this region and the mass M_- in \mathcal{M}_- . For the second asymmetric solution of the pair these masses are interchanged because the two are related by reflection. The same holds for the particle number.

When the mass of the three solutions is then considered versus their respective particle number, it becomes clear, that asymmetric solutions are more strongly bound than the symmetric solutions. Thus the spontaneous symmetry breaking leads to energetically favoured configurations.

All of this remains valid when gravity is coupled. However, gravity introduces further new features. First of all, the backreaction of the boson field on the metric removes the lower frequency bound ω_{\min} . For small values of the coupling constant κ a spiralling behaviour arises, that is known from compact stars. However, unlike for compact stars, here the spirals unwind again. We attribute this effect to the presence of the negative energy density in the form of a phantom field, since such unwinding has, for instance, also been observed for bosonic configurations in Einstein-Gauß-Bonnet theory [27].

Second, the presence of gravity generates a further bifurcation phenomenon at ω_{crg} . Here the pair of asymmetric solutions merges again with the symmetric ones. Only the symmetric solutions then persist to small boson frequencies. Third, we observe a transition in the geometry of the solutions. For small κ this transition arises in the vicinity of the spiral. Here the single throat solutions develop an equator and a second throat. For the symmetric solutions, the equator is then localized at the radial coordinate $\eta = 0$, surrounded symmetrically by both throats. For the asymmetric solutions, in contrast, the equator and the second throat originate far from the first throat in the other part of the manifold.

Here we have performed most of the calculations in five spacetime dimensions. However, the formalism is general for D dimensions. Pure Ellis wormholes in D dimensions have been obtained in [13]. It should be straightforward to include bosonic matter and obtain the analogues of the configurations studied here also in $D > 5$ dimensions. In particular, we expect that the phenomenon of spontaneous symmetry breaking will be present independent of the dimension.

We have already shown that this phenomenon is also present in four dimensions. In this case, such configurations of bosonic matter surrounding wormholes might also be of potential astrophysical interest [12]. In particular, one can obtain solutions which differ widely in mass and size by varying the potential for the complex scalar field. For instance, there may also be solutions which mimic compact astrophysical objects like neutron stars or black holes. These solutions could be studied in the context of gravitational lensing [28–30], with respect to their light curves [22], their geodesics [31], etc.

Let us conclude with some remarks on the stability of these solutions. The Ellis wormholes are known to be unstable [13, 32–34]. The unstable radial mode of the wormholes was shown to persist in the presence of bosonic matter for symmetric solutions in four dimensions [12], where the instability was weakened by the presence of matter.

Now, that we have seen, that spontaneous symmetry breaking occurs, we conjecture that the symmetric solutions will acquire another unstable mode from the matter side. In contrast, the asymmetric solutions will only possess the single unstable wormhole mode (on their fundamental branch). While we defer a full analysis of the stability of the symmetric and asymmetric solutions to a later time, we note already, that the analysis of the solutions in the probe limit precisely conforms to this expectation. Here, at the bifurcation frequencies ω_{crI} and ω_{crU} the symmetric solutions indeed exhibit a zero mode, which then turns into a second unstable mode in the parameter space where the asymmetric solutions exist. For the asymmetric solutions stability could possibly be achieved by removing the phantom field and modifying gravity instead [35–43].

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V. APPENDIX

We briefly explain, how we obtain the mass and the particle number of the asymmetric solutions. For these special care is needed, since there is some ambiguity as to where to put the lower limit of the respective volume integrals. We therefore extract these global charges from the asymptotic behaviour of the solutions. For the symmetric solutions there is no such ambiguity.

A. Mass in the probe limit

In the probe limit the backreaction of the matter on the spacetime is not taken into account, since the matter equation is solved in the background of the Ellis wormhole. Then the mass cannot be extracted from the asymptotic form of the metric. To obtain the mass anyway asymptotically, we resort to the following construction.

We consider the Einstein equation

$$R_0^0 \sqrt{-g} = \kappa \left(T_0^0 + \frac{1}{2-D} T_\mu^\mu \right) \sqrt{-g} \quad (39)$$

and treat the metric function a as order $\mathcal{O}(\kappa)$. Consequently, only the background metric enters the right hand side. Evaluation yields

$$-\frac{D-3}{2} \left(\left[(ph)^{\frac{D-2}{2}} \frac{1}{\sqrt{p}} \right] a' \right)' = \kappa \left(T_0^0 + \frac{1}{2-D} T_\mu^\mu \right) (ph)^{\frac{D-2}{2}} \sqrt{p}. \quad (40)$$

Defining $\rho_m = \left[(ph)^{\frac{D-2}{2}} \frac{1}{\sqrt{p}} \right] a'$ we find

$$\rho_m(\eta) = -\frac{2}{D-3} \kappa \int_{-\infty}^{\eta} \left(T_0^0 + \frac{1}{2-D} T_\mu^\mu \right) (ph)^{\frac{D-2}{2}} \sqrt{p} d\eta' + \rho_{m0}, \quad (41)$$

where ρ_{m0} is an integration constant. In the next step we integrate $a' = \left[(ph)^{\frac{2-D}{2}} \sqrt{p} \right] \rho_m$,

$$\begin{aligned} a(\eta) = & -\frac{2\kappa}{D-3} \int_{-\infty}^{\eta} \left[(ph)^{\frac{2-D}{2}} \sqrt{p} \right] \left\{ \int_{-\infty}^{\eta'} \left(T_0^0 + \frac{1}{2-D} T_\mu^\mu \right) (ph)^{\frac{D-2}{2}} \sqrt{p} d\eta'' \right\} d\eta' \\ & + \rho_{m0} \int_{-\infty}^{\eta} \left[(ph)^{\frac{2-D}{2}} \sqrt{p} \right] d\eta' + a_0, \end{aligned} \quad (42)$$

where a_0 is again an integration constant. Next we find the integration constants ρ_{m0} and a_0 from the boundary conditions $a(\pm\infty) = 0$,

$$\rho_{m0} = \frac{2\kappa}{D-3} \frac{\int_{-\infty}^{\infty} \left[(ph)^{\frac{2-D}{2}} \sqrt{p} \right] \left\{ \int_{-\infty}^{\eta} \left(T_0^0 + \frac{1}{2-D} T_\mu^\mu \right) (ph)^{\frac{D-2}{2}} \sqrt{p} d\eta' \right\} d\eta}{\int_{-\infty}^{\infty} \left[(ph)^{\frac{2-D}{2}} \sqrt{p} \right] d\eta}, \quad a_0 = 0. \quad (43)$$

The masses M_\pm are related to the asymptotic behaviour of the function a ,

$$M_\pm = \pm \frac{D-3}{2\kappa} [\eta^{D-2} a']_{\pm\infty} \Omega_{D-2} = \pm \frac{D-3}{2\kappa} \rho_m(\pm\infty) \Omega_{D-2}. \quad (44)$$

Explicitly,

$$\begin{aligned} M_+ &= \left\{ -\frac{D-2}{D-3} \int_{-\infty}^{\infty} \left(T_0^0 + \frac{1}{2-D} T_\mu^\mu \right) (ph)^{\frac{D-2}{2}} \sqrt{p} d\eta + \frac{D-2}{2\kappa} \rho_{m0} \right\} \Omega_{D-2}, \\ M_- &= -\frac{D-2}{2\kappa} \rho_{m0} \Omega_{D-2}. \end{aligned} \quad (45)$$

We note that M_+ can be written as

$$M_+ = \frac{D-2}{D-3} \int_{\Sigma} \left(T_{\mu\nu} + \frac{1}{2-D} g_{\mu\nu} T_{\lambda}^{\lambda} \right) n^{\mu} \xi^{\nu} dV + \frac{D-2}{2\kappa} \rho_{m0} \Omega_{D-2} , \quad (46)$$

where Σ denotes a spacelike hypersurface including both asymptotic regions of \mathcal{M}_+ and \mathcal{M}_- .

B. Particle number via electric charge

We consider a fictitious electrostatic potential $\Phi_{\text{el}}(\eta)$ sourced by the current j^{μ} , Eq. (30),

$$\partial_{\mu} (\sqrt{-g} g^{\mu\nu} g^{tt} \partial_{\nu} \Phi_{\text{el}}) = -j^t \sqrt{-g} . \quad (47)$$

For the spherically symmetric Ansatz this yields

$$\left(e^{(D-3)a} (ph)^{\frac{D-2}{2}} \frac{1}{\sqrt{p}} \Phi'_{\text{el}} \right)' = j^t e^{-a} (ph)^{\frac{D-2}{2}} \sqrt{p} . \quad (48)$$

The charges Q_{\pm} can then be obtained from

$$Q_{\pm} = \mp \left[\eta^{D-2} \Phi'_{\text{el}} \right]_{\pm\infty} \Omega_{D-2} . \quad (49)$$

Introducing the auxiliary quantity $\rho_q = e^{(D-3)a} (ph)^{\frac{D-2}{2}} \frac{1}{\sqrt{p}} \Phi'_{\text{el}}$ we obtain from Eq. (48)

$$\rho_q(\eta) = \int_{-\infty}^{\eta} j^t e^{-a} (ph)^{\frac{D-2}{2}} \sqrt{p} d\eta' + \rho_{q0} \quad (50)$$

with integration constant ρ_{q0} . On the other hand, expressing Φ'_{el} in terms of ρ_q and integrating yields

$$\Phi_{\text{el}}(\eta) = \int_{-\infty}^{\eta} e^{-(D-3)a} (ph)^{-\frac{D-2}{2}} \sqrt{p} \left[\int_{-\infty}^{\eta'} j^t e^{-a} (ph)^{\frac{D-2}{2}} \sqrt{p} d\eta'' \right] d\eta' + \rho_{q0} \int_{-\infty}^{\eta} e^{-(D-3)a} (ph)^{-\frac{D-2}{2}} \sqrt{p} d\eta' + \Phi_{\text{el}0} , \quad (51)$$

where $\Phi_{\text{el}0}$ is another integration constant. In order to determine the integration constants ρ_{q0} and $\Phi_{\text{el}0}$ we impose the boundary conditions $\Phi_{\text{el}}(\pm\infty) = 0$. This yields $\Phi_{\text{el}0} = 0$ and

$$\rho_{q0} = - \frac{\int_{-\infty}^{\infty} e^{-(D-3)a} (ph)^{-\frac{D-2}{2}} \sqrt{p} \left[\int_{-\infty}^{\eta} j^t e^{-a} (ph)^{\frac{D-2}{2}} \sqrt{p} d\eta' \right] d\eta}{\int_{-\infty}^{\infty} e^{-(D-3)a} (ph)^{-\frac{D-2}{2}} \sqrt{p} d\eta} . \quad (52)$$

We observe that $Q_{\pm} = \mp \rho_q(\pm\infty) \Omega_{D-2}$. Consequently,

$$Q_+ = - \left(\int_{-\infty}^{\infty} j^t e^{-a} (ph)^{\frac{D-2}{2}} \sqrt{p} d\eta + \rho_{q0} \right) \Omega_{D-2} , \quad Q_- = \rho_{q0} \Omega_{D-2} . \quad (53)$$

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- [1] H. G. Ellis, J. Math. Phys. **14**, 104 (1973).
 - [2] H. G. Ellis, Gen. Rel. Grav. **10**, 105 (1979).
 - [3] K. A. Bronnikov, Acta Phys. Polon. **B4**, 251 (1973).
 - [4] M. S. Morris, K. S. Thorne, Am. J. Phys. **56**, 395 (1988).
 - [5] M. S. Morris, K. S. Thorne and U. Yurtsever, Phys. Rev. Lett. **61**, 1446 (1988).
 - [6] C. Armendariz-Picon, Phys. Rev. **D65**, 104010 (2002).
 - [7] S. V. Sushkov, Phys. Rev. D **71**, 043520 (2005).
 - [8] F. S. N. Lobo, Phys. Rev. D **71**, 084011 (2005).
 - [9] V. Dzhunushaliev, V. Folomeev, B. Kleihaus and J. Kunz, JCAP **1104**, 031 (2011).
 - [10] V. Dzhunushaliev, V. Folomeev, B. Kleihaus and J. Kunz, Phys. Rev. D **85**, 124028 (2012).
 - [11] E. Charalampidis, T. Ioannidou, B. Kleihaus, and J. Kunz, Phys. Rev. D **87**, 084069 (2013).

- [12] V. Dzhunushaliev, V. Folomeev, C. Hoffmann, B. Kleihaus and J. Kunz, Phys. Rev. D **90**, 124038 (2014).
- [13] T. Torii and H. a. Shinkai, Phys. Rev. D **88**, 064027 (2013).
- [14] V. Dzhunushaliev, V. Folomeev, B. Kleihaus, J. Kunz and E. Radu, Phys. Rev. D **88**, 124028 (2013).
- [15] B. Kleihaus and J. Kunz, Phys. Rev. D **90**, 121503 (2014).
- [16] X. Y. Chew, B. Kleihaus and J. Kunz, Phys. Rev. D **94**, 104031 (2016).
- [17] R. Friedberg, T. D. Lee and A. Sirlin, Phys. Rev. D **13**, 2739 (1976).
- [18] R. Friedberg, T. D. Lee, Y. Pang, Phys. Rev. **D35**, 3658 (1987)
- [19] S. R. Coleman, Nucl. Phys. B **262**, 263 (1985) [Erratum-ibid. B **269**, 744 (1986)].
- [20] T. D. Lee, Y. Pang, Phys. Rept. **221**, 251 (1992).
- [21] O. Hauser, R. Ibadov, B. Kleihaus and J. Kunz, Phys. Rev. D **89**, 064010 (2014)
- [22] V. Dzhunushaliev, V. Folomeev, B. Kleihaus and J. Kunz, Phys. Rev. D **89**, 084018 (2014).
- [23] R. M. Wald, “General Relativity”, (University of Chicago Press, Chicago, 1984).
- [24] U. Ascher, J. Christiansen, R. D. Russell, A collocation solver for mixed order systems of boundary value problems, Mathematics of Computation **33**, 659 (1979);
U. Ascher, J. Christiansen, R. D. Russell, Collocation software for boundary-value ODEs, ACM Transactions **7**, 209 (1981).
- [25] B. Hartmann, B. Kleihaus, J. Kunz and M. List, Phys. Rev. D **82**, 084022 (2010).
- [26] M. S. Volkov and E. Wöhnert, Phys. Rev. D **66**, 085003 (2002).
- [27] B. Hartmann, J. Riedel and R. Suci, Phys. Lett. B **726**, 906 (2013).
- [28] F. Abe, Astrophys. J. **725**, 787 (2010).
- [29] Y. Toki, T. Kitamura, H. Asada and F. Abe, Astrophys. J. **740**, 121 (2011).
- [30] R. Takahashi and H. Asada, Astrophys. J. **768**, L16 (2013).
- [31] V. Diemer, K. Eilers, B. Hartmann, I. Schaffer and C. Toma, Phys. Rev. D **88**, 044025 (2013)
- [32] H. -a. Shinkai and S. A. Hayward, Phys. Rev. D **66**, 044005 (2002).
- [33] J. A. Gonzalez, F. S. Guzman, and O. Sarbach, Class. Quant. Grav. **26**, 015011 (2009).
- [34] J. A. Gonzalez, F. S. Guzman, and O. Sarbach, Class. Quant. Grav. **26**, 015010 (2009).
- [35] D. Hochberg, Phys. Lett. **B251**, 349 (1990).
- [36] H. Fukutaka, K. Tanaka, K. Ghoroku, Phys. Lett. **B222**, 191 (1989).
- [37] K. Ghoroku, T. Soma, Phys. Rev. **D46**, 1507 (1992).
- [38] N. Furey, A. DeBenedictis, Class. Quant. Grav. **22**, 313 (2005).
- [39] K. A. Bronnikov and E. Elizalde, Phys. Rev. D **81**, 044032 (2010).
- [40] P. Kanti, B. Kleihaus and J. Kunz, Phys. Rev. Lett. **107**, 271101 (2011).
- [41] P. Kanti, B. Kleihaus and J. Kunz, Phys. Rev. D **85**, 044007 (2012).
- [42] F. S. N. Lobo and M. A. Oliveira, Phys. Rev. D **80**, 104012 (2009).
- [43] T. Harko, F. S. N. Lobo, M. K. Mak and S. V. Sushkov, Phys. Rev. D **87**, 067504 (2013).